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ON ASSOCIATIVE ALGEBRAS SATISFYING THE ENGEL CONDITION

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ABSTRACT
It is shown that every finitely generated associative algebra over a field of characteristic $p > 0$ satisfying the Engel condition is Lie-nilpotent. It follows that the Engel condition is inherited from an algebra A to its group of units, $U(A)$.

In Lie algebras over a field of characteristic zero, the Engel condition

$$[x, \underbrace{y, y, \ldots, y}_{n}] = 0$$

implies nilpotency. This remarkable result, recently proved by Zel'manov [Ze], fails to hold in positive characteristic (see [Co], [Ra]). Yet, a celebrated theorem of Kostrikin ([Ko1], [Ko2]), proved in the context of the restricted Burnside problem, states that, if $n \leqq p$, then every n-Engel Lie algebra over a field of characteristic p is locally nilpotent. This was extended by A. Braun for $n = p + 1$ [Br1]. For many years it was not known whether the local nilpotency follows without the restriction on n. Recently it was announced that this has been settled in the affirmative by Zel'manov, in his general solution to the restricted Burnside problem.

The purpose of this note is to show that the Engel condition implies nilpotency in Lie algebras of characteristic p, arising from finitely generated associative algebras. Since these need not be finitely generated as Lie algebras,

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this result is not included in Zel'manov's (as yet unpublished) work. The proof, which is surprisingly short, uses some machinery of the theory of PI-algebras, such as Braun's theorem on the nilpotency of the Jacobson radical, and Shirshov's solution to the Kurosch problem. We also need a theorem of K. W. Gruenberg on the nilpotency of finitely generated soluble Engel Lie algebras.

We denote the Lie-powers of L by $L^{[i]}$ (to avoid confusion with associative powers). We use the terms Lie-nilpotent/Lie-soluble, instead of nilpotent/soluble, when referring to the Lie-structure of associative rings. From now on we assume that p is a fixed prime, and K is a field of characteristic p.

THEOREM. *Let A be a finitely generated associative algebra over K. Suppose A satisfies the Engel condition. Then A is Lie-nilpotent.*

PROOF. The proof is divided into 7 short stages.

(1) Let $J(A)$ be the Jacobson radical of A. Then $A/J(A)$, as a semiprimitive PI-algebra, satisfies the same identities as some matrix ring $M_k(C)$ (where C is commutative). But $M_k(C)$ satisfies the Engel condition only when $k = 1$. Therefore $A/J(A)$ is commutative. This classical argument is due to Amitsur (see [Am] for stronger results of this type).

(2) Since A is an affine PI-algebra, it follows by Braun's theorem [Br2] that $J(A)$ is a nilpotent ideal. Let A' be the ideal of A generated by all the commutators $[x, y]$. Then $A' \subseteq J(A)$ (by (1)), so that A' is also nilpotent.

(3) If $\{\delta_i(A)\}_{i \geq 0}$ is the derived series of A as a Lie algebra, then $\delta_{i+1}(A) \subseteq (A')^{2^i}$. We conclude that A is Lie-soluble.

(4) Choose a positive integer i, such that A is p^i-Engel. Then, since $\mathrm{char}(K) = p$, we have $\mathrm{ad}(x^{p^i}) = \mathrm{ad}(x)^{p^i} = 0$. This means that $x^{p^i} \in Z(A)$ (the center of A), for all $x \in A$. In particular, A is integral over $Z(A)$.

(5) Since A is a PI finitely generated $Z(A)$-algebra, it follows from Shirshov's solution to the Kurosch problem (see [Ro, Cor. 4.2.9]) that A is finitely generated as a $Z(A)$-module.

(6) Suppose $A = \Sigma_{i=1}^{m} x_i \cdot Z(A)$. Let L be the Lie sub-algebra of A, generated by $x_1, \ldots, x_m$. Then, by (3), L is soluble. Since L is also finitely generated, it follows by [Gr, Thm. 1′] that L is nilpotent.

(7) Clearly $A = L \cdot Z(A)$. Therefore, assuming $L^{[t]} = 0$, we obtain $A^{[t]} = L^{[t]} \cdot Z(A)^t = 0$, so that A itself is Lie-nilpotent. □

REMARK. Applying the theorem for the free n-Engel K-algebra on d generators, we obtain a quantitative version, namely: If A is an associative n-

Engel K-algebra with d generators, then $A^{[f(n,d)]} = 0$, for some fixed function f. This will be used in the sequel.

We now consider the group-theoretical version of the Engel identity

$$(x, \underbrace{y, y, \ldots, y}_{n}) = 1, \qquad \text{where } (x, y) \text{ denotes group-commutator.}$$

COROLLARY. *Let A be an associative algebra over K, and let $U(A)$ be its group of units. Suppose A is n-Engel. Then $U(A)$ is m-Engel, for some m depending on n.*

PROOF. For $x, y \in U(A)$, consider the finitely generated associative subalgebra $B = K\{x, y, x^{-1}, y^{-1}\} \subseteq A$. By the previous result, $B^{[t]} = 0$, for some t depending on n ($t = f(n, 4)$, in our notation). According to a theorem of Gupta and Levin [GL], if $\{\gamma_j\}_{j \geqq 1}$ denotes the lower central series of $U(B)$, then, for all $j \geqq 1$,

$$\gamma_j \subseteq 1 + B^{[j]}.$$

In particular, $\gamma_t = 1$. But $x, y \in U(B)$. Therefore

$$(x, \underbrace{y, y, \ldots, y}_{t-1}) = 1.$$

Since x, y are arbitrary, it follows that $U(A)$ is $(t-1)$-Engel. □

REMARK. The proof also shows that $U(A)$ is locally nilpotent, and in fact satisfies the following stronger property: every d-generator subgroup of $U(A)$ has nilpotency class $\leqq g(n, d)$, for some function g. It should be noted that, in the zero characteristic case, $U(A)$ itself is nilpotent (since A is Lie-nilpotent, by Zelmanov's theorem).

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